Aldo Montesano PRINCIPLES OF ECONOMIC ANALYSIS

CHAPTER 6 CHOICE OVER TIME

The preceding analysis of consumer's choice and producer's choice has already included intertemporal aspects since goods are defined also with respect to the time of their availability. For example, the choice of the production can take into account an input which is available at time t and an output which is available at time t+2, so that the difference between the two times, equal to 2, indicates the time necessary for production. In this chapter we will highlight the main aspects of choice over time. The term *dynamic*, which is sometimes used, means that the choice is represented by a function of time. It is convenient to distinguish between two cases: one in which there is only one choice concerning the goods available at different points of time, and one that consists of a sequence of choices at different points of time. In the first case, there is only one choice that has as its object a function of time (which indicates the stream of consumption). In the second case, there is a sequence of choices and, therefore, the function of time that represents this sequence is related to the points of time when the decisions are taken. Nevertheless, this distinction is often disregarded because of some hypotheses that make these two different cases identical. By identical we mean that in both of the cases the same quantities of goods are chosen. When that happens, we say that there is *temporal dynamic consistency.*

An example of temporal inconsistency is represented by the history of Ulysses and the sirens (illustrated by Strotz, 1956). Ulysses can choose between listening and not listening to the sirens singing. He knows that if he listens to them he might not be able to resist their charm and may not be able to run away from them in which case they will kill him. He himself would like to listen to the sirens but not let them kill him (to do so, he asks to be tied to the mast of the boat and plugs the ears of the other sailors that are on the boat with him). In this story, there are two distinct points in time: before listening to the sirens and after that. The preferences of Ulysses, if the choice he makes is unique, between the three alternatives (listening and going to the sirens is denoted by a; listening and not going to the sirens by a'; and not listening by a'') are $a' \succ a'' \succ a$. If there is a sequence of choices, the alternatives are: in the first period of time between listening, a_1, and not listening, a_1', and if the first choice was a_1, there is the second period where he can choose between two alternatives: going to the sirens, a_2, and not going to the sirens, a_2'. In this example, the preferences in the second period are $a_2 \succ a_2'$. Since the preferences in the first period are (a_1 and a_2') $\succ a_1' \succ$ (a_1 and a_2) (these preferences coincide with the preferences $a' \succ a'' \succ a$ that were stated earlier), it is best to choose in the first period the alternative a_1'

(in this way he prevents from getting the worst outcome, because he knows that if in the first period he chose the other option, that is a_1, he would choose a_2 in the second period). The temporal dynamic inconsistency in this example comes from the fact that in the first case, when there is only one unique choice, he chooses a' (listening to the sirens, but not going to them), while in the second case, when there is a sequence of choices, he chooses a_1' (not listening), so different timing specifications result in different behavior. A careful reader will note that the temporal dynamic inconsistency can occur only if preferences at some periods of time depend on the choices made in previous periods. He will also note that the example we just described would represent well the issues of addiction (for example to drugs). The trick that Ulysses makes is to transform the problem with a sequence of choices into the problem with only one choice. Doing so, he can enforce his most preferred outcome. Unfortunately, such a transformation is not always possible (what would Ulysses do if he could not tie himself to the mast or if he could not make his sailors temporarily deaf?).

In the remaining of this chapter, we will examine only the case of the unique choice, calling it *intertemporal choice*, by introducing the analytical instruments which are easier to use. The most important aspect that characterizes this type of choice is the dependence of the set of possible actions at some point of time on the actions taken in the previous periods. In other words, the agent chooses in the first period (t = 0) a (finite or infinite) sequence of actions $a_0, a_1, a_2, ..., a_T$ taking into account that the action in period t, that is a_t, can be chosen from the set B_t, where B_t is a function of the actions taken in the preceding periods, i.e. $B_t(a_0, a_1, ..., a_{t-1})$ for $t > 0$, whilst for $t = 0$ there is a given set of possible actions B_0. The object of the intertemporal choice is dynamic, because it is represented by a function of time, and yet the intertemporal choice is static, in the sense that there is only one unique choice to be made and not a sequence of choices at different points of time. Consequently, the analysis indicated in Chapter 2 applies to the intertemporal choice: the *intertemporal action* is $a = (a_0, a_1, a_2, ..., a_T)$, which is a vector of temporary actions, the set of possible actions is $B = \{(a_0, a_1, a_2, ..., a_T) : a_0 \in B_0 \text{ and } a_t \in B(a_0, a_1, ..., a_{t-1}), t = 1, ..., T\}$ and the choice set is $S(B) \subseteq B$.

The intertemporal choice is particularly interesting under uncertainty. The content of this chapter, in which the uncertainty is absent, should nevertheless be very useful as an introduction to the analysis of choice under uncertainty.

6.1 Consumer's intertemporal preferences and choice

Let's consider intertemporal choice of consumption and assume that the system of preferences of the consumer is represented by an ordinal utility function $U(x_0, x_1, ..., x_T)$, where $x_t \in \mathbb{R}_+^k$ represents the consumption in period t (bear in mind that, if the availability of the goods in time is explicitly indicated, then the qualitative characteristics that define the good do not include the time in which the good is available) and T is the time horizon of the consumer (it is a positive finite or infinite integer number).

There are some assumptions that are commonly imposed on the utility function. They simplify the mathematical analysis and at the same time are an approximate representation of the true preferences. We assume that the system of preferences can be represented by a utility function that has the following features (where the second one implies the first one).

The first assumption on the utility function is *additive separability*. We assume that the utility has the following propriety

$$U(x_0, x_1, x_2, ..., x_T) = \sum_{t=0}^{T} u_t(x_t)$$

In other words, the *intertemporal utility* U is the sum of the *temporary utilities* u_t. This assumption is not satisfied if, for example, the sign of the difference $U(x_0, x_1', ..., x_T') - U(x_0, x_1, ..., x_T)$ depends on x_0, which could occur if the timing of the consumption plays a role in determining consumer's preferences.

The second assumption imposes *stationarity* on the temporary utilities, but for a subjective constant *discount factor*. Therefore, it assumes that

$$U(x_0, x_1, x_2, ..., x_T) = \sum_{t=0}^{T} \delta^t u(x_t)$$

where $\delta \in (0, 1]$ is the discount factor. This assumption is not satisfied for example for goods, like *windsurfing*, for which preferences depend on the age of the decision maker. Notice that this function is independent from the time of the choice (but for the discount factor, which does not influence one's choice). That is, letting h denote the time of the choice and U^h the corresponding intertemporal utility, we have $U^h(x_h, ..., x_T) = \sum_{t=0}^{T-h} \delta^t u(x_{h+t})$ and $U^0(x_h, ..., x_T) = \sum_{t=0}^{T-h} \delta^{h+t} u(x_{h+t})$, so that $U^h(x_h, ..., x_T) = \delta^h U^0(x_h, ..., x_T)$.

Finally, we assume that the utility function is strictly increasing (that is, $u(x_t') > u(x_t)$ if $x_t' > x_t$) and concave.

The typical example of an intertemporal choice is the choice of the consumption stream during lifetime given the income flow $I_0, I_1, ..., I_T$ (non negative).

For T = 0, we have the following problem $\max_{x_0 \in B(p_0, I_0)} u(x_0)$, where $B(p_0, I_0) = \left\{x_0 \in \mathbb{R}_+^k \colon p_0 x_0 \le I_0\right\}$, which we have already studied in Chapter 3.

For T = 1, assuming that the agent can borrow or lend money at an interest rate i, we have the following problem $\max_{(x_0, x_1) \in B(p_0, p_1, I_0, I_1)} u(x_0) + \delta u(x_1)$,

where $B(p_0, p_1, I_0, I_1) = \left\{ x_0, x_1 \in \mathbb{R}^k_+ : p_0 x_0 + \frac{p_1 x_1}{1+i} \le I_0 + \frac{I_1}{1+i} \right\}$, since the sum $S_1 = I_0 - p_0 x_0$ saved (if positive) or borrowed (if negative) determines the second period budget constraint $p_1 x_1 \le I_1 + (1+i) S_1$. The constraint $p_0 x_0 + \frac{p_1 x_1}{1+i} \le I_0 + \frac{I_1}{1+i}$, called *intertemporal budget constraint*, requires that the present value of spending is not higher than the present value of income flow. (If it was impossible to transfer wealth across time periods, we would have two constraints $p_0 x_0 \le I_0$ and $p_1 x_1 \le I_1$. In this case, the intertemporal choice would coincide with two separate temporary problems $\max_{x_0 \in B(p_0, I_0)} u(x_0)$ and $\max_{x_1 \in B(p_1, I_1)} u(x_1)$. If it was possible to lend but not to borrow, we would have the following constraint
$B(p_0, p_1, I_0, I_1) = \left\{ x_0, x_1 \in \mathbb{R}^k_+ : p_0 x_0 \le I_0 \text{ and } p_0 x_0 + \frac{p_1 x_1}{1+i} \le I_0 + \frac{I_1}{1+i} \right\}$).

Keeping in mind that the monotonicity of the utility function implies that the budget constraint is satisfied with equality, we can introduce the following Lagrangian function

$$\mathrm{L}(x_0, x_1, \lambda) = u(x_0) + \delta u(x_1) - \lambda \left(p_0 x_0 + \frac{p_1 x_1}{1+i} - I_0 - \frac{I_1}{1+i} \right)$$

which yields the following first order conditions, that describe the internal solution (that is, for $x_0^* >> 0$ and $x_1^* >> 0$),

$$\mathrm{D}u(x_0^*) = \lambda^* p_0,$$

$$\delta \mathrm{D}u(x_1^*) = \lambda^* \frac{p_1}{1+i},$$

$$p_0 x_0^* + \frac{p_1 x_1^*}{1+i} = I_0 + \frac{I_1}{1+i}.$$

The second order conditions are satisfied under the assumption of concavity of the temporary utility function $u(x_t)$.

If the temporary utility function is strongly monotone and $p_0 = p_1 >> 0$, so that $(1+i)\delta \mathrm{D}u(x_1^*) = \mathrm{D}u(x_0^*)$, then $x_1^* >> x_0^*$ if $\delta > 1/(1+i)$ (that is, if the discount factor, that measures the patience of the agents by indicating the amount of utility at time 0 equivalent to one unit of utility at time 1, is larger than the interest rate, that measures the present value of one unit of cash at the next date) and $x_1^* << x_0^*$ if $\delta < 1/(1+i)$. Therefore, if $I_0 = I_1$, $p_0 = p_1 >> 0$ and $\delta > 1/(1+i)$, we have that $I_1 < p_1 x_1^* > p_0 x_0^* < I_0$, while, if $\delta < 1/(1+i)$, we have that $I_1 > p_1 x_1^* < p_0 x_0^* > I_0$.

Interesting observations about the choice of savings $S_1^* = I_0 - p_0 x_0^*$ can be made with relation to the comparative static analysis of the choice

(x_0*, x_1*) characterized before. In particular, we can determine the impact of the interest rate i on the savings S_1* by looking at the derivative $\frac{\partial S_1 *}{\partial i}$. Keeping the assumptions made earlier, we infer that this derivative is non negative for the component that constitutes the substitution effect; and negative if the savings S_1* are positive (and positive if the savings S_1* are negative) for the component that constitutes income effect (analogous to the description in Paragraph 4.5). Therefore, the resulting savings do not necessarily have to be increasing in the interest rate: they can be decreasing if the savings are positive and the income effect dominates over the substitution effect.

The proof can be obtained by using the Slutsky equation presented in Paragraph 4.5. Since $I_0 = p_0\omega_0$, $I_1 = p_1\omega_1$, $z_0 = x_0 - \omega_0$, $z_1 = x_1 - \omega_1$, $z = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ and $p = \begin{bmatrix} p_0 \\ p_1/(1+i) \end{bmatrix}$, the budget constraint $p_0 x_0 + \frac{p_1 x_1}{1+i} \le I_0 + \frac{I_1}{1+i}$ becomes $pz \le 0$ and we obtain the problem $\max_{pz\le 0} u(z_0+\omega_0) + \delta u(z_1+\omega_1)$, which is a particular case of the problem described in Paragraph 4.5. Since $S_1 = -p_0 z_0$, we have that $\frac{\partial S_1 *}{\partial i} = -p_0{}^T \frac{\partial e_0(p)}{\partial i} = -p_0{}^T \mathrm{D}_p e_0(p) \begin{bmatrix} p_0 \\ 0 \end{bmatrix}$, where $e(p) = \arg\max_{pz\le 0} u(z_0+\omega_0) + \delta u(z_1+\omega_1)$, with $e(p) = \begin{bmatrix} e_0(p) \\ e_1(p) \end{bmatrix}$. The Slutsky equation described in Paragraph 4.5 requires that $\mathrm{D}_p e(p) = S(p) - \mathrm{D}_m e(p,m) e(p)^T$, where $S(p)$ is the substitution matrix and $m = I_0 + I_1/(1+i)$. Since $e_0(p) = \begin{bmatrix} I & 0 \end{bmatrix} e(p)$, where I and 0 are respectively identity and zero matrix (both $k \times k$), we have $\mathrm{D}_p e_0(p) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{D}_p e(p)$, for which $\mathrm{D}_p e_0(p) = \begin{bmatrix} I & 0 \end{bmatrix} S(p) - \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{D}_m e(p,m) e(p)^T$ and so $\frac{\partial S_1 *}{\partial i} = -\begin{bmatrix} p_0{}^T & 0^T \end{bmatrix} S(p) \begin{bmatrix} p_0 \\ 0 \end{bmatrix} - p_0{}^T \mathrm{D}_m e_0(p,m) S_1 *$. Since the substitution matrix is negative semidefinite, we have that $-\begin{bmatrix} p_0{}^T & 0^T \end{bmatrix} S(p) \begin{bmatrix} p_0 \\ 0 \end{bmatrix} \ge 0$, so the substitution effect is non negative. The income effect is negative (positive) if $S_1* > 0$ ($S_1* < 0$) assuming that the consumption for the first period is normal, that is $p_0{}^T \mathrm{D}_m e_0(p,m) > 0$.

The previous problem of the intertemporal choice with $T = 1$ can be solved in stages, rather than in just one stage. We start from the last period and first consider the following problem $\max_{x_1 \in B_1(S_1, p_1, I_1)} u(x_1)$, where $B_1(S_1, p_1, I_1) = \left\{ x_1 \in \mathbb{R}_+^k : p_1 x_1 \le I_1 + (1+i) S_1 \right\}$. The solution determines the *demand correspondence* $x_1(S_1, p_1, I_1)$ and the indirect utility function

$u_1*(S_1,p_1,I_1)=\max_{x_1\in B_1(S_1,p_1,I_1)} u(x_1)$ $u(x_1)$, called (*maximum*) *value function*. This is a problem that the consumer would encounter in period $t = 1$ given that he saved S_1 in the first period. Therefore, the choice in $t = 0$ has to take into account the impact of S_1 on the intertemporal utility. As a result, in period $t=0$ we have the problem $\max_{(x_0,S_1)\in B_0(p_0,I_0)} u(x_0)+\delta u_1*(S_1,p_1,I_1)$ where $B_0(p_0,I_0)=\left\{x_0\in\mathbb{R}_+^k, S_1\in\mathbb{R}: p_0x_0+S_1=I_0\right\}$. The solution to this problem determines the consumption in the first period x_0 and savings S_1, that in turn, through the demand correspondence $x_1(S_1,p_1,I_1)$, determine x_1, that is the consumption in the next period.

The problems we analyze can be represented by the following Lagrangian functions

$$\mathrm{L}_1(x_1,\lambda_1)=u(x_1)-\lambda_1\left(p_1x_1-I_1-(1+i)S_1\right),$$

$$\mathrm{L}_0(x_0,S_1,\lambda_0)=u(x_0)+\delta u_1*(S_1,p_1,I_1)-\lambda_0\left(p_0x_0+S_1-I_0\right)$$

which, accordingly, give rise to the following first order conditions

$$\mathrm{D}u(x_1*)=\lambda_1*p_1, \qquad p_1x_1*=I_1+(1+i)S_1*$$

and

$$\mathrm{D}u(x_0*)=\lambda_0*p_0, \qquad \delta\frac{\partial u_1*}{\partial S_1}=\lambda_0*, \qquad p_0x_0*+S_1*=I_0.$$

Keeping in mind that $\frac{\partial u_1*}{\partial S_1}=\lambda_1*(1+i)$ (Proposition 3.11), we get that $\lambda_0*=\lambda_1*\delta(1+i)$ and, therefore, these first order conditions coincide with the ones in the original problem (in which we did not divide the problem into two stages and call $\lambda*$ the multiplier λ_0*),

The approach that we used in the case with two periods, that is when $T=1$, can be extended to a case with any number of periods, $T > 1$. Denoting with p, x and I, respectively, the vectors $(p_0,p_1,...,p_T)$, $(x_0,x_1,...,x_T)$ and $(I_0,I_1,...,I_T)$, we obtain the intertemporal problem

$$\max_{x\in B(p,I)}\sum_{t=0}^{T}\delta^t u(x_t), \text{ where } B(p,I)=\left\{x\in\mathbb{R}_+^{k(T+1)}: \sum_{t=0}^{T}\frac{p_tx_t}{(1+i)^t}\le\sum_{t=0}^{T}\frac{I_t}{(1+i)^t}\right\}.$$

From the Lagrangian function

$$\mathrm{L}(x,\lambda)=\sum_{t=0}^{T}\delta^t u(x_t)-\lambda\left(\sum_{t=0}^{T}\frac{p_tx_t}{(1+i)^t}-\sum_{t=0}^{T}\frac{I_t}{(1+i)^t}\right)$$

we obtain the following first order conditions

$$\delta^t\mathrm{D}u(x_t*)=\lambda*p_t(1+i)^{-t} \text{ for } t=0,1,...,T, \qquad \sum_{t=0}^{T}\frac{p_tx_t}{(1+i)^t}=\sum_{t=0}^{T}\frac{I_t}{(1+i)^t}.$$

This problem can also be solved in stages, rather than at just one stage, starting from the last period T (if the number of periods T is finite).

Then, for $t = T$, we have the problem $\max_{x_T \in B_T(S_T, p_T, I_T)} u(x_T)$, where $B_T(S_T, p_T, I_T) = \left\{x_T \in \mathbb{R}^k_+ : p_T x_T \le I_T + (1+i)S_T\right\}$. The solution to this problem defines the demand correspondence $x_T(S_T, p_T, I_T)$ and the (maximum) value function $u_T{}^*(S_T, p_T, I_T) = \max_{x_T \in B_T(S_T, p_T, I_T)} u(x_T)$.

For $t = T-1, T-2, ..., 1$, we have to solve the sequence of problems

$$\max_{(x_t, S_{t+1}) \in B_t(S_t, p_t, I_t)} u(x_t) + \delta u_{t+1}{}^*(S_{t+1}, p_{t+1}, I_{t+1})$$

where $B_t(S_t, p_t, I_t) = \left\{x_t \in \mathbb{R}^k_+, S_t \in \mathbb{R} : p_t x_t + S_{t+1} = I_t + (1+i)S_t\right\}$. Solving these problems we obtain $x_t(S_t, p_t, I_t)$, $S_{t+1}(S_t, p_t, I_t)$ and $u_t{}^*(S_t, p_t, I_t) = \max_{(x_t, S_{t+1}) \in B_t(S_t, p_t, I_t)} u(x_t) + \delta u_{t+1}{}^*(S_{t+1}, p_{t+1}, I_{t+1})$. At the last stage, which concerns the first period, we have the problem $\max_{(x_0, S_1) \in B_0(p_0, I_0)} u(x_0) + \delta u_1{}^*(S_1, p_1, I_1)$, where $B_0(p_0, I_0) = \left\{x_0 \in \mathbb{R}^k_+, S_1 \in \mathbb{R} : p_0 x_0 + S_1 = I_0\right\}$, that determines the consumption x_0 and savings S_1 in the first period. These in turn, through the demand correspondence $x_1(S_1, p_1, I_1)$ and the savings function $S_2(S_1, p_1, I_1)$, determine the consumption in the next period x_1 and the total savings S_2. In the same manner, for each $t = 1, 2, ..., T$, through $x_t(S_t, p_t, I_t)$ and $S_{t+1}(S_t, p_t, I_t)$ we obtain the corresponding consumption levels.

For all these problems we can write down Lagrangians

$$\mathrm{L}_t(x_t, S_{t+1}, \lambda_t) = u(x_t) + \delta u_t{}^*(S_{t+1}, p_{t+1}, I_{t+1}) - \lambda_t\left(p_t x_t + S_{t+1} - I_t - (1+i)S_t\right), \quad t = 0, 1, ..., T,$$

with $S_0 = S_{T+1} = 0$ and $u_{T+1}{}^*(S_{T+1}, p_{T+1}, I_{T+1}) = 0$, that give rise to the following first order conditions

$$\mathrm{D}u(x_t{}^*) = \lambda_t{}^* p_t, \qquad p_t x_t{}^* + S_{t+1}{}^* = I_t + (1+i)S_t{}^*, \qquad \text{for } t = 0, 1, ..., T,$$

$$\delta \frac{\partial u_{t+1}{}^*}{\partial S_{t+1}} = \lambda_t{}^*, \qquad \text{for } t = 0, 1, ..., T-1.$$

Recalling that $\frac{\partial u_{t+1}{}^*}{\partial S_{t+1}} = \lambda_{t+1}{}^*(1+i)$, we obtain $\lambda_t{}^* = \lambda_{t+1}{}^*\delta(1+i)$ and consequently $\lambda_t{}^* = \lambda_0{}^*\delta^{-t}(1+i)^{-t}$. Therefore, with $\lambda^* = \lambda_0{}^*$, we find out that these first order conditions coincide with those of the original problem (that did not divide the problem into stages).